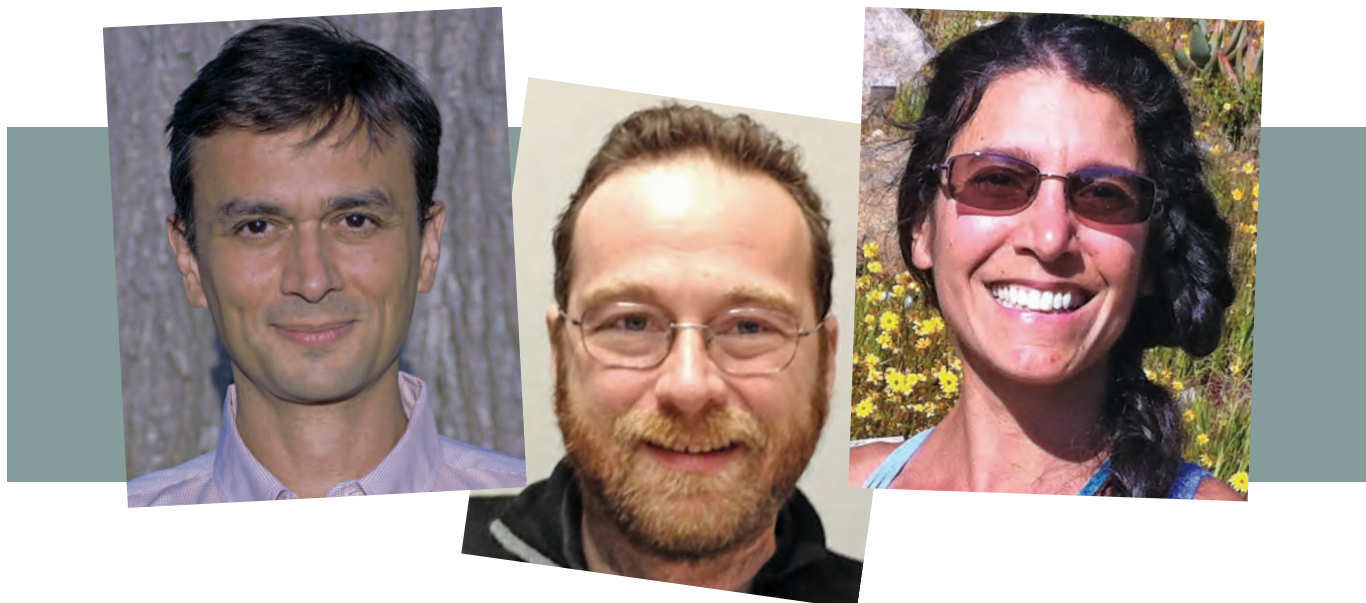


AMS FALL WESTERN SECTIONAL SAMPLER



From left to right: Paul Balmer, Pavel Etingof, Monica Vazirani.

In this sampler, the speakers above have kindly provided introductions to their Invited Addresses for the upcoming AMS Fall Western Sectional Meeting.

November 4–5, 2017
(Saturday–Sunday)
University of California, Riverside
Riverside, CA

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DOI: <http://dx.doi.org/10.1090/noti1591>



An Invitation to Tensor-Triangular Geometry

by Paul Balmer

Tensor-triangulated categories appear in topology, in algebraic geometry, in representation theory, and in the theory of C^* -algebras for instance. The aim of tensor-triangular geometry is to develop a unified treatment of this broad variety of examples. By way of introduction to this advanced topic, let us discuss an example in homotopy theory based on joint work [2] with Beren Sanders.

As a warm-up, consider a finite group G acting on a finite set X . For every subgroup H of G we can count the number of fixed points $|X^H|$ of X under the action of H . This number depends on H only up to conjugacy in G . Moreover, if p is a prime number, then $|X^H|$ is congruent modulo p to the number $|X^K|$ for any normal subgroup $K < H$ of index p . For illustration, consider the extreme case of the cyclic group $G = C_p$ of order p , with $K = 1$ trivial and $H = G$. What we are saying is that the number of elements in the finite set X is congruent modulo p to the number of fixed points $|X^G|$; indeed the difference $X - X^G$ consists of G -orbits which all have p elements. For a general group G , if one can move from the subgroup H to the subgroup K via a tower of index- p normal subgroups and conjugation in G , then the number $|X^H|$ will be congruent modulo p to the number $|X^K|$ independently of the chosen G -set X .

Building on the above basic combinatorics, Dress [3] described the Zariski spectrum of prime ideals of the so-called *Burnside ring* $A(G)$. One can add and multiply finite G -sets by taking disjoint union and product. The commutative ring $A(G)$ is obtained from this by formally including opposites for addition. Taking H -fixed points defines a ring homomorphism $f^H : A(G) \rightarrow \mathbb{Z}$ from the Burnside ring to the ring of integers $\mathbb{Z} = A(1)$. Pulling back ordinary prime ideals of \mathbb{Z} via f^H , we obtain prime ideals in $A(G)$. Dress's theorem states that this construction catches all the prime ideals of the Burnside ring. The congruence discussed in our warm-up now becomes the following fact: If H and K are related as above (by towers of index- p subgroups and conjugacy) for some prime p , then the prime ideal of the Burnside ring pulled back from the prime $p\mathbb{Z}$ via f^H coincides with the one pulled back via f^K . This collision is the only redundancy among primes in the Burnside ring in terms of primes of \mathbb{Z} and conjugacy classes of subgroup H . For instance, the spectrum of $A(C_p)$ is given in Figure 1, in which $p(H, q)$ denotes the pulled-back prime $(f^H)^{-1}(q\mathbb{Z})$ for the two available subgroups $H = 1$ and $H = C_p$. The green dots form the copy of $\text{Spec}(\mathbb{Z})$ pulled back via f^1 ("forget the action"), and the red dots form the copy of $\text{Spec}(\mathbb{Z})$ pulled back via the G -fixed points homomorphism f^G . The collision happens here at a single green-red point.

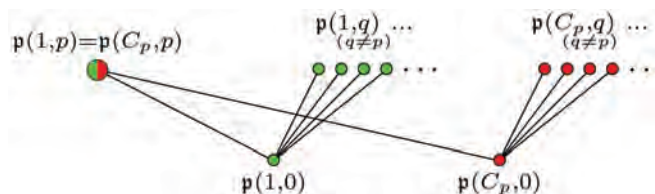


Figure 1. The Zariski spectrum of the Burnside ring $A(C_p)$. The pullbacks collide at the single green-red point.

Let us now ask a similar question, replacing finite G -sets with topological spaces, on which our finite group G acts continuously. The problem has thus gained immense complexity due to the wild class of such available G -spaces. To handle this complexity, we pass to suitable stable homotopy categories. In doing so, the prime ideals of commutative algebra will be replaced by a sophisticated notion of primes. In the nonequivariant case, that is, for $G = 1$, Hopkins and Smith [4], building on earlier work with Devinatz, established the celebrated *chromatic filtration* in stable homotopy theory. Consider the Spanier-Whitehead category SH^c of finite pointed CW-complexes, up to homotopy and up to stabilization, i.e., after formally inverting suspension. This SH^c is a *tensor-triangulated* category; i.e., it comes with a tensor product (smash product) and with exact triangles (Puppe sequences). Given a prime number p , one can localize this Spanier-Whitehead category at p by inverting all other primes. This produces the p -local stable homotopy category of finite spectra. Hopkins and Smith's theorem asserts that finite p -local spectra are essentially classified by a number, the so-called (p -local) *chromatic height*, which can be defined in terms of Morava K -theories. The chromatic height of X is greater than or equal to the chromatic height of Y if and only if X can be constructed out of Y by means of the available tensor-triangular operations: cones, direct summands, suspension, etc.

In tensor-triangular geometry one considers prime subcategories of a tensor-triangulated category T in an analogous way to the prime ideals of a commutative ring in commutative algebra. Taken together, these tensor-triangular primes form a space, $\text{Spc}(T)$, called the tensor-triangular spectrum of T . The topological version of Dress's result for the Burnside ring can now be phrased in terms of the tensor-triangular spectrum of $\text{SH}(G)^c$, the tensor-triangulated category of finite G -spectra. The latter is a G -equivariant analogue of the category SH^c studied by Hopkins and Smith. For every subgroup H of G , we have a geometric H -fixed points functor F^H from $\text{SH}(G)^c$ to SH^c , and very much as in the Dress case we can pull back tensor-triangular primes from SH^c to $\text{SH}(G)^c$ via F^H and obtain G -equivariant tensor-triangular primes denoted $P(H, p, n)$ for every subgroup H , prime number p , and chromatic height n . Our theorem [2] is that these $P(H, p, n)$ are all the tensor-triangular primes of $\text{SH}(G)^c$. Moreover, the collision observed by Dress is resolved, in that $P(H, p, n)$ and $P(K, p, n)$ are now distinct when $K < H$ is a normal

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DOI: <http://dx.doi.org/10.1090/noti1594>

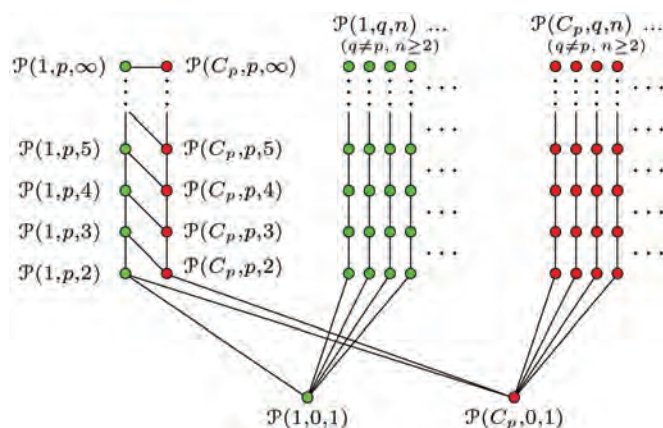


Figure 2. The tensor-triangular spectrum of $\mathrm{SH}(C_p)^c$. The point $P(1, p, n)$ does not equal $P(C_p, p, n)$ but does at least sit in the closure of the point $P(C_p, p, n-1)$.

subgroup of index p . However, something survives of the Dress collision, in that the point $P(K, p, n+1)$ now sits in the closure of the point $P(H, p, n)$. The spectrum $\mathrm{Spc}(\mathrm{SH}(G)^c)$ is illustrated for the special case of $G = C_p$ in Figure 2. Here the green and red dots are the images of the Hopkins–Smith chromatic primes under F^1 (forget the action) and F^G (geometric fixed points) respectively.

From the determination of the tensor-triangular spectrum, one can always classify all objects up to the available tensor-triangular operations. The resulting *equivariant chromatic filtration* in the case of $\mathrm{SH}(G)^c$ can also be found in [2]. Such a classification of all objects up to the available tensor-triangular operations is the best classification one can hope for, because most tensor-triangulated categories we come across are wild. These ideas go way beyond (equivariant) homotopy theory, for example, to algebraic geometry, representation theory, and the theory of motives. The aim of tensor-triangular geometry is to develop a unified treatment of this broad variety of examples. For an introduction, see my article on “Tensor triangular geometry” [1].

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Figures 1 and 2 courtesy of Paul Balmer.

Photo of Paul Balmer courtesy of P. Croset.



Paul Balmer

ABOUT THE AUTHOR

Paul Balmer is working on the development of “tensor-triangular geometry,” an umbrella topic that touches parts of algebraic geometry, stable homotopy theory, and modular representation theory.

Double Affine Hecke Algebras and Their Applications

by Pavel Etingof

I am very excited to have been asked to deliver an invited address at the Fall 2017 meeting of the AMS Western Section (UC Riverside). I will talk about double affine Hecke algebras and their applications.

It is well known that studying the representation theory of various algebraic structures is a rich source for unifying a number of parts of mathematics. Double affine Hecke algebras are a particular type of algebraic structure whose representation theory has recently become quite important. They were discovered twenty-five years ago by I. Cherednik as a tool for proving the Macdonald conjectures about orthogonal polynomials attached to root systems. But since then it has become clear that they have a much broader meaning. Nowadays they not only play a central role in representation theory but also have numerous connections to many other fields—integrable systems, quantum groups, knot theory, algebraic geometry, and combinatorics, for example.

Double affine Hecke algebras are often defined by a long list of relations, but these relations actually have a simple meaning in terms of elementary topology. Namely, let Σ be a connected orientable 2-dimensional surface, and let $C_n(\Sigma)$ be the configuration space of n -tuples of distinct unlabelled points on Σ , i.e., the complement in Σ^n/S_n of the loci where some of the points coincide. The fundamental group $B_n(\Sigma) := \pi_1(C_n(\Sigma))$ is called *the braid group of Σ* . This group contains an element T that corresponds to two of the points exchanging their positions. More precisely, since we haven’t fixed a base point, this element is defined only up to conjugation. However, the quotient group $B_n(\Sigma)/(T^2)$ of $B_n(\Sigma)$ by the relation $T^2 = 1$ is well-defined. By the Seifert–van Kampen theorem, quotienting by this relation corresponds to gluing back into $C_n(\Sigma)$ the loci where some points coincide, which gives the orbifold Σ^n/S_n . Thus, $B_n(\Sigma)/(T^2)$ is the orbifold fundamental group of this orbifold; i.e., it is isomorphic to $S_n \ltimes \pi_1(\Sigma)^n$.

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DOI: <http://dx.doi.org/10.1090/noti1592>

Now let us pass from groups to group algebras over some commutative base ring k (for example, $k = \mathbb{C}$). Then the relation $T^2 = 1$ may be rewritten as $T^2 - 1 = 0$ or $(T - 1)(T + 1) = 0$. Thus, given $t_1, t_2 \in k$ (usually assumed to be invertible), we may consider a deformation of this relation:

$$(1) \quad (T - t_1)(T + t_2) = 0.$$

Definition 1. The *Hecke algebra* of Σ , $H_{t_1, t_2}(\Sigma)$ is the quotient of the group algebra $k[B_n(\Sigma)]$ by relation (1).

One can show that unless $\Sigma = S^2$ is the 2-sphere, this gives a flat deformation of $k[S_n \ltimes \pi_1(\Sigma)^n]$ (in the formal sense).

For example, when Σ is the 2-plane then $H_{t_1, t_2}(\Sigma)$ is the finite Hecke algebra, and when Σ is the cylinder $H_{t_1, t_2}(\Sigma)$ is the affine Hecke algebra (in these cases, the algebra actually depends only on $t := t_1/t_2$). This motivates the following definition.

Definition 2. The *double affine Hecke algebra* is the algebra $H_{t_1, t_2}(\mathbf{T})$, where \mathbf{T} is the 2-torus.

Here the word “double” refers to the fact that the torus has two independent 1-cycles.

The algebra $H_{t_1, t_2}(\mathbf{T})$ is thus a flat deformation of $k[S_n \ltimes \mathbb{Z}^{2n}]$, and, unlike the previous two examples, it now genuinely depends on both parameters t_1, t_2 .

In a similar way one can define the double affine Hecke algebra of a finite crystallographic reflection group (i.e., Weyl group) W with reflection representation V . If $W = S_n$ and $V = \mathbb{C}^n$, this recovers the algebra $H_{t_1, t_2}(\mathbf{T})$ defined above.

In my talk, I will discuss the basic properties of double affine Hecke algebras and touch upon some applications.

ABOUT THE AUTHOR

Pavel Etingof's research focuses on double affine Hecke algebras, quantum groups, and tensor categories. He also runs PRIMES, a research program in mathematics, engineering, and science at MIT.



Pavel Etingof

Combinatorics, Categorification, and Crystals

by Monica Vazirani

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DOI: <http://dx.doi.org/10.1090/noti1593>

ABSTRACT. Categorification attempts to replace sets or algebraic and geometric structures with more general categories. It has enjoyed amazing successes, such as Khovanov homology categorifying the Jones polynomial knot invariant, KLR algebras categorifying quantum groups, or Soergel bimodules categorifying Hecke algebras. Many of the algebras we see in categorification can be described diagrammatically. This is related to a historic motivation for categorification: to construct knot and link invariants. The payoffs to finding these richer structures include not only constructing finer knot invariants, but proving positivity results and producing some fantastic mathematics.

In this article, we will focus on the second example, that is, on quantum groups as well as their crystal bases or canonical bases, which exhibit the positivity and integrality that is a trademark feature of a decategorified structure.

In combinatorics, we love to see phenomena that exhibit integrality and positivity. It means there is something interesting to count. It is special for positive integers to occur as structure constants or change of basis coefficients or the coefficients of a polynomial or power series. When this happens it is a bit of an art form to discover a combinatorial structure that realizes these numbers. Once discovered, there can be unexpected payoffs. Further, it can turn computing such coefficients into a fun game, such as counting all the ways to rearrange tiles of a given shape on a board.

The program of categorification is another art form that has enjoyed amazing successes. It is an art form because the process of categorification is not at all well defined. Just as in combinatorics we seek to replace a positive integer with an interesting set of that cardinality, in categorification sets and functions should be replaced with categories and functors. But the process need not start with the category \mathbf{Sets} , and it is easier to provide a precise definition of the reverse notion, called “decategorification.” In the classical setting, decategorification is a well-defined process that turns isomorphic objects of a category into objects that are equal. In a more general setting, decategorification includes passing from a category \mathcal{C} to its Grothendieck group $G_0(\mathcal{C})$ or its split Grothendieck group $K_0(\mathcal{C})$ or its Trace. In geometric settings, homology theories and cohomology theories are also examples of decategorification. The space that arises from decategorifying a particular category \mathcal{C} may have interesting algebraic structure that results from the structure of \mathcal{C} . For example, the decategorification may be an abelian group, a ring, a Hopf algebra, or the representation space of some other algebra, as in some of the examples above.

Let us first consider the category $\mathbf{Vect}_{\mathbb{C}}$ of finite-dimensional vector spaces, which we decategorify by taking the Grothendieck group $G_0(\mathbf{Vect}_{\mathbb{C}})$. $G_0(\mathbf{Vect}_{\mathbb{C}})$ is the \mathbb{Z} -module generated by isomorphism classes of vector spaces, with the relation $[B] = [A] + [C]$ whenever $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact. Hence $G_0(\mathbf{Vect}_{\mathbb{C}}) \simeq \mathbb{Z}$. In fact, this is a ring isomorphism, as $\mathbf{Vect}_{\mathbb{C}}$ has the operations of direct sum and tensor product, which descend to $+$ and \times in \mathbb{Z} . (But we will continue to call $G_0(\mathcal{C})$ the Grothendieck

group even when endowed with other structures.) Hence a categorification of \mathbb{Z} is $\mathcal{C} = \mathcal{Vect}_{\mathbb{C}}$, and in the process we lift the integer 5 to a 5-dimensional vector space.

Combinatorialists have been categorifying for decades, even if not thinking of it in that way, often taking \mathcal{C} to be \mathcal{Sets} —wonderfully interesting (finite) sets that look like puzzle pieces. Decategorification here sends a set to its isomorphism class, which we can represent by its cardinality, as two sets are isomorphic exactly when they have the same cardinality. Hence it is easy to decategorify and extract the number 5 from $\{(4), (31), (2^2), (21^2), (1^4)\}$, the set of integer partitions of 4. But it is not so straightforward to reverse this. Note that 5 can also be realized as the number of standard Young tableaux of shape (32) or the number of ways to triangulate a pentagon or the size of a basis of our favorite 5-dimensional vector space or indeed the number of elements in any set of cardinality 5.

Next, consider $\mathcal{C} = \mathbb{C}[\mathfrak{S}_4]\text{-mod}$, or, better yet, let $\mathcal{C} = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[\mathfrak{S}_n]\text{-mod}$, which has induction and restriction functors that let us pass between different \mathbb{N} -graded pieces. Its Grothendieck group is isomorphic to the ring (or Hopf algebra) of symmetric functions. In other words, we say the symmetric group categorifies symmetric functions. More precisely, $\mathbb{C}[\mathfrak{S}_4]\text{-mod}$ decategorifies to the space of degree 4 symmetric functions, which we could further decategorify to the set of partitions of 4. Further, the spaces have distinguished bases: the 5 simple $\mathbb{C}[\mathfrak{S}_4]\text{-modules}$ correspond to the 5 Schur functions of degree 4, which are both naturally indexed by the 5 partitions of 4.

Additionally, refined induction and restriction functors decategorify to linear operators that give the graded vector space $\bigoplus_{n \in \mathbb{N}} \mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathbb{C}[\mathfrak{S}_n]\text{-mod})$ the structure of a highest weight representation $V(\Lambda_0)$ of the Lie algebra \mathfrak{sl}_{∞} or of its enveloping algebra $U(\mathfrak{sl}_{\infty})$. In other words, symmetric groups also categorify $V(\Lambda_0)$, sometimes known as Fock space. The isomorphism classes of simple modules form a distinguished (weight) basis of $V(\Lambda_0)$. As multiplicities of simple modules in a composition series are nonnegative integers, this basis enjoys wonderful positivity properties. It is a crystal basis, a dual Lusztig canonical basis, and a Kashiwara upper global crystal basis. (Dually, one can work with indecomposable projectives for the split Grothendieck group $K_0(\mathcal{C})$.) We can even replace \mathbb{C} with \mathbb{F}_p , and \mathfrak{sl}_{∞} with $\hat{\mathfrak{sl}}_p$, and then the simples yield a dual p -canonical basis, which has even stronger positivity properties than the dual canonical basis in this case. The known integrality and positivity properties of crystal bases and canonical bases as well as their combinatorial structure were a hint that $V(\Lambda_0)$ could be categorified. We understood that Young's lattice of partitions described the combinatorics of the symmetric group $(\mathbb{C}[\mathfrak{S}_n]\text{-mod})$ as well as that of $V(\Lambda_0)$ before we knew why.

The above story raises the question of whether we can replace $U(\mathfrak{sl}_{\infty})$ with the quantum Kac-Moody algebra $U_q(\mathfrak{g})$ of type X , as this is also a setting in which we have crystal bases and canonical bases. In groundbreaking work Khovanov-Lauda and Rouquier construct \mathcal{C} as the

category of modules over the KLR algebra, or quiver Hecke algebra, of arbitrary symmetrizable type X . I have heard Kleshchev describe KLR algebras as being type A (like the symmetric group \mathfrak{S}_n) but in characteristic X . For example, characteristic $\hat{\mathfrak{sl}}_p$ behaves like the more familiar characteristic p , but characteristic E_8 also makes sense. I love this description.

The KLR algebra is graded, which introduces the quantum parameter q . We replace $n \in \mathbb{N}$ with $\nu \in Q^+$, the positive cone in the root lattice; replace \mathfrak{S}_n with the graded algebra $R(\nu)$; and replace $\bigoplus_{n \in \mathbb{N}} \mathbb{C}[\mathfrak{S}_n]$ with $R = \bigoplus_{\nu \in Q^+} R(\nu)$. Then $\mathcal{C} = R\text{-mod}$ categorifies $U_q^+(\mathfrak{g})$, and certain quotients called cyclotomic KLR algebras categorify highest weight representations $V(\lambda)$ for dominant integral weights $\lambda \in P^+$. Then we recover the crystal graph $B(\lambda)$ via the simple R -modules.

The crystal graph $B(\lambda)$ is a combinatorial skeleton of the irreducible highest weight representation $V(\lambda)$ of $U_q(\mathfrak{g})$. In “the $q \rightarrow 0$ limit” $V(\lambda)$ has a crystal basis that comprises the nodes of the graph $B(\lambda)$. The graph has other data, including directed colored edges that are roughly limits of an integral form of raising operators.

Over $\mathfrak{g} = \mathfrak{sl}_{\infty}$ the crystal graph $B(\Lambda_0)$ looks like Young's lattice of partitions with extra decoration. Although the μ -weight space of $V(\lambda)$ can have high multiplicity, a node of $B(\lambda)$ can be a source (or sink) of at most one edge of given color. In this way, the crystal knows how to “break ties” and fits into a multiplicity-free setting, much like the branching rule for the symmetric group over \mathbb{C} is multiplicity-free. Crystal graphs are one of my favorite combinatorial tools. I like to believe that in hindsight the blueprints to the KLR algebras were dictated by the crystal graphs. Their generators and relations are precisely what they need to be for restriction functors to have the right multiplicity-free properties and to yield the appropriate subcrystals in rank 2. Beautiful combinatorics can be harvested from this program of categorification, but (to mix a few metaphors) I also think we glimpse combinatorial shadows before we know what is casting them, and we should keep an eye out for these blueprints.

Photo Credit

Photo of Monica Vazirani courtesy of Dorothy Buck.



Monica Vazirani

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Monica Vazirani's research is in combinatorial representation theory.